1. Obtain mathematical models \( \frac{X(s)}{U(s)} \) of the mechanical systems shown below.

Solution

a)
Here, the input is \( u(t) \) and the output is the displacement \( x \) as shown in the figure.

\[
u(t) = m\ddot{x} + kx
\]

the rollers under the mass means there is no friction.

Converting the above equation to Laplace domain.

\[
U(s) = ms^2X(s) + kX(s)
\]

Taking \( X(s) \) out of the brackets:

\[
U(s) = (ms^2 + k)X(s)
\]

Then the transfer function (T.F) equal to:

\[
\frac{X(s)}{U(s)} = \frac{1}{ms^2 + k}
\]

b)

The solution will be the same procedure, but the spring factor will be:

\[
k_T = \frac{k_1k_2}{k_1 + k_2}
\]

Then the T.F would be:

\[
\frac{X(s)}{U(s)} = \frac{1}{ms^2 + k_T}
\]

Note:

The second derivative of \( x \) could be presented as:

\[
\frac{d^2}{dt^2} \text{ or } \ddot{x} \text{ or } x''
\]
A–3–1. Figure 3–20(a) shows a schematic diagram of an automobile suspension system. As the car moves along the road, the vertical displacements at the tires act as the motion excitation to the automobile suspension system. The motion of this system consists of a translational motion of the center of mass and a rotational motion about the center of mass. Mathematical modeling of the complete system is quite complicated.

A very simplified version of the suspension system is shown in Figure 3–20(b). Assuming that the motion $x_i$ at point $P$ is the input to the system and the vertical motion $x_o$ of the body is the output, obtain the transfer function $X_o(s)/X_i(s)$. (Consider the motion of the body only in the vertical direction.) Displacement $x_o$ is measured from the equilibrium position in the absence of input $x_i$.

Solution. The equation of motion for the system shown in Figure 3–20(b) is

$$m\ddot{x}_o + b(\dot{x}_o - \dot{x}_i) + k(x_o - x_i) = 0$$

or

$$m\ddot{x}_o + bx_o + kx_o = b\dot{x}_i + kx_i$$

Taking the Laplace transform of this last equation, assuming zero initial conditions, we obtain

$$(ms^2 + bs + k)X_o(s) = (bs + k)X_i(s)$$

Hence the transfer function $X_o(s)/X_i(s)$ is given by

$$\frac{X_o(s)}{X_i(s)} = \frac{bs + k}{ms^2 + bs + k}$$

**Figure 3–20**

(a) Automobile suspension system; (b) simplified suspension system.
A–3–2. Obtain the transfer function $Y(s)/U(s)$ of the system shown in Figure 3–21. The input $u$ is a displacement input. (Like the system of Problem A–3–1, this is also a simplified version of an automobile or motorcycle suspension system.)

Solution. Assume that displacements $x$ and $y$ are measured from respective steady-state positions in the absence of the input $u$. Applying the Newton’s second law to this system, we obtain

$$m_1\ddot{x} = k_2(y - x) + b(\dot{y} - \dot{x}) + k_1(u - x)$$
$$m_2\ddot{y} = -k_2(y - x) - b(\dot{y} - \dot{x})$$

Hence, we have

$$m_1\ddot{x} + b\dot{x} + (k_1 + k_2)x = b\dot{y} + k_2y + k_1u$$
$$m_2\ddot{y} + b\dot{y} + k_2y = b\dot{x} + k_2x$$

Taking Laplace transforms of these two equations, assuming zero initial conditions, we obtain

$$[m_1s^2 + bs + (k_1 + k_2)]X(s) = (bs + k_2)Y(s) + k_1U(s)$$
$$[m_2s^2 + bs + k_2]Y(s) = (bs + k_2)X(s)$$

Eliminating $X(s)$ from the last two equations, we have

$$(m_1s^2 + bs + k_1 + k_2)\frac{m_2s^2 + bs + k_2}{bs + k_2}Y(s) = (bs + k_2)Y(s) + k_1U(s)$$

which yields

$$\frac{Y(s)}{U(s)} = \frac{k_1(bs + k_2)}{m_1m_2s^3 + (m_1 + m_2)bs^3 + [k_1m_2 + (m_1 + m_2)k_2]s^2 + k_1bs + k_1k_2}$$

![Figure 3–21](image)

Suspension system.
2. **Obtain the transfer function** \( E_0(s)/E_i(s) \) **of the electrical circuit shown** in.

![Electrical Circuit Diagram]

**The equations for the given circuit are as follow:**

\[
R_1 L_1 + L\left(\frac{di_1}{dt} - \frac{di_2}{dt}\right) = e_i
\]

\[
R_2 i_2 + \frac{1}{c} \int i_2 dt + L\left(\frac{di_2}{dt} - \frac{di_1}{dt}\right) = 0
\]

\[
\frac{1}{c} \int i_2 dt = e_0
\]

**The Laplace transforms of these three equations, with zero initial conditions, are**

\[
R_1 I_1(s) + L[I_1(s) - I_2(s)] = E_i(s) \quad (1)
\]

\[
R_2 I_2(s) + \frac{1}{cs} I_2(s) + L[I_2(s) - I_1(s)] = 0 \quad (2)
\]

\[
\frac{1}{cs} I_2(s) = E_0(s) \quad (3)
\]

**From Equation (2) we obtain.**

\[
\left(R_2 + \frac{1}{cs} + Ls\right)I_2(s) = LsI_1(s)
\]

**or**
\[ I_2(s) = \frac{LCs^2}{LCs^2 + R_2Cs + 1} I_1(s) \] (4)

Or

Substituting equation (4) into equation (1), we get

\[(R_1 + LS - LS \frac{LCs^2}{LCs^2 + R_2Cs + 1})I_1(s) = E_i(s)\]

Or

\[
\frac{LC(R_1 + R_2)s^2 + (R_1R_2C + L)s + R_1}{LCs^2 + R_2Cs + 1} I_1(s) = E_1(s)
\] (5)

From Equation (3) and (4), we get

\[
\frac{LS}{LCs^2 + R_2Cs + 1} I_1(s) = E_0(s)
\] (6)

From equation (5) and (6), we obtain

\[
\frac{E_0(s)}{E_i(s)} = \frac{LS}{LC(R_1 + R_2)s^2 + (R_1R_2C + L)s + R_1}
\]

3. Consider the system shown in Figure below. An armature-controlled dc servomotor drives a load consisting of the moment of inertia \( J_L \). The torque developed by the motor is \( T \). The moment of inertia of the motor rotor is \( J_m \). The angular displacements of the motor rotor and the load element are \( \theta_m \) and \( \theta \), respectively. The gear ratio is \( n = \theta/\theta_m \). Obtain the transfer function \( \Theta(s)/E_i(s) \).

\[ \text{Solution} \]

Define the current in the armature circuit to be \( i_a \) then, we have
\[ L \frac{di_a}{dt} + Ri_a + K_b \frac{d\theta_m}{dt} = e_i \]

Or

Taking Laplace transforms

\[(Ls + R)i_a(s) + K_b s\Theta_m(s) = E_i(s) \quad (1)\]

Where \(K_b\) is the back emf constant of the motor. We also have

\[ J_m \ddot{\theta}_m + T = T_m = Ki_a \quad (2) \]

\[ T = \frac{\theta}{\theta_m}T_L = nT_L \]

\[ J_L \ddot{\theta} = T_L \]

Where \(K\) is the motor torque constant and \(i_a\) is the armature current. Equation (2) can be rewritten as.

\[(J_m + n^2J_L)\dddot{\theta} = nKi_a \quad \text{where } \dddot{\theta}_m = \dddot{\theta}/n \]

Or

Taking Laplace transforms

\[(J_m + n^2J_L)s^2\Theta(s) = nKI_a(s) \quad (3)\]

By substituting equation (3) into equation (1), we obtain.

\[ (Ls + R)\left(\frac{J_m + n^2J_L}{nK}\right)s^2\Theta(s) + K_b s \frac{\Theta(s)}{n} = E_i(s) \]

Or

\[ [(Ls + R)(J_m + n^2J_L)s^2\Theta(s) + KK_b s] = nKE_i(s) \]
Hence

\[
\frac{\theta(s)}{E_i(s)} = \frac{nK}{[(Ls + R)(J_m + n^2J_L)s^2\theta(s) + KK_b)s]}
\]